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Non-holonomic Lagrangian systems in jet manifolds

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Abstract. A geometrical setting in terms of jet manifolds is developed for time-dependent non-holonomic Lagrangian systems. An almost product structure on the evolution space is constructed in such a way that the constrained dynamics is obtained by projection of the free dynamics. A constrained Poincaré–Cartan 2-form is defined. If the non-holonomic system is singular, a constraint algorithm is constructed. Special attention is devoted to Čaplygin systems and a reduction theorem is proved.

1. Introduction

In a recent paper [19] (see also [14, 15, 17, 18]), we have developed a geometrical setting for non-holonomic time-independent Lagrangian systems, where the constraints are linear on the velocities. That is, the Lagrangian function is $L = L(q^A, \dot{q}^A)$ and the typical constraint functions are of the form $\phi_i(q^A, \dot{q}^A) = (\mu_i)_A(q)\dot{q}^A$.

The aim of the present paper is to extend that geometrical framework for the case of Lagrangian systems given by a time-dependent Lagrangian function $L = L(t, q^A, \dot{q}^A)$ and constraint functions which are affine on the velocities, say $\phi_i(t, q^A, \dot{q}^A) = (\mu_i)_A(t, q)\dot{q}^A + h_i(t, q)$. It seems almost evident that, in order to globalize the picture, we need to use affine bundles [5, 8, 16, 21, 22]. In fact, the geometrical setting is as follows. We start with a fibration $\pi : E \longrightarrow \mathbb{R}$ and, then, we take the 1-jet prolongation $J^1\pi$, which is, in fact, an affine bundle over E modelled on the vector bundle $V\pi$. So, the Lagrangian function is supposed to be defined on $J^1\pi$ (the evolution space) and the constraints are obtained as the evaluation maps of a local cobasis of a distribution D on E. It should be remarked that a compatibility condition with the fibration has to be assumed on D in order to obtain independent constraint functions as the theory demands in the classical setting [3, 4, 25].

Our approach leads us to write the constrained motion equations in an intrinsic way, without explicit mention of Lagrange multipliers. To do this, we lift D to two new distributions on $J^1\pi$. A regularity condition on the constrained system is assumed to obtain a solution of the dynamics. The regularity condition is automatically satisfied for Lagrangian functions which are positive or negative definite, a usual assumption in the

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literature. In the regular case, we define an almost product structure $(\mathcal{P}, \mathcal{Q})$ on $J^1\pi$ along the constraint submanifold $\tilde{D}$ such that the dynamics is just the projection by $\mathcal{P}$ of the solution of the unconstrained system.

One of the main results of this paper is the following. There exists a constrained Poincaré–Cartan 2-form $\tilde{\omega}$ on $\tilde{D}$ such that the solution of the dynamics is a unique non-autonomous second-order differential equation living in its kernel. The result could be interesting for quantization purposes, as we will show in a forthcoming paper. We notice that the constrained Poincaré–Cartan 2-form $\tilde{\omega}$ coincides (up to the sign) with the one obtained by Sarlet, Cantrijn and Saunders [27–29, 31].

If the constrained system is not regular, we construct a constraint algorithm which gives a final constraint submanifold $\tilde{D}_f$ of $\tilde{D}$ on where there is a solution of the dynamics. Of course, the dynamics is no longer unique. The procedure is quite similar to that developed by Gotay and Nester [9–11] for singular Lagrangians. The constrained submanifolds are obtained by demanding the preservation of the constraints on the time, as in the Dirac–Bergmann formalism [7].

The Hamiltonian counterpart is also studied. Nothing special is obtained since both formalisms are 'isomorphic' by means of the Legendre transformation. However, the results illustrate the differences in comparison with the time-independent case.

A special kind of constrained system is studied at the end of the paper, the so-called Čaplygin systems. They are constrained systems where the constraints are imposed by the existence of a connection in some intermediate fibration $E \longrightarrow N \longrightarrow \mathbb{R}$. In other words, the motions have to be horizontal curves. We assume that the Lagrangian function is invariant by horizontal lifts. This is just the case when we are in presence of principal fibrations and we demand invariance by the action of the structure group [12]. We obtain a sort of reduction procedure which remembers the symplectic reduction procedure. In fact, our procedure gives a reduced free Lagrangian subjected to a non-conservative force in such a way that the original dynamics are obtained by horizontal lift of the reduced one. We can say that for non-holonomic systems the invariance by connections plays the same role that the invariance by symmetries does for unconstrained systems. This reduction procedure permits us to relate the constants of motion for the reduced system with the ones for the original constrained system.

2. Evolution spaces

Let E be an $(n+1)$-dimensional fibred manifold over $\mathbb{R}$, i.e., there exists a surjective submersion

$$\pi : E \longrightarrow \mathbb{R}.$$

We denote by $J^1\pi$ the 1-jet manifold of local sections of π, namely

$$J^1\pi = \left\{ \begin{array}{c} j_t^1\phi/\phi : U \subset \mathbb{R} \longrightarrow E, \pi \circ \phi = \mathrm{id}_U \\ U \text{ open neighbourhood of } t \end{array} \right\}.$$

If (t, q^A) are fibred coordinates on E, then $J^1\pi$ has local coordinates (t, q^A, v^A). In fact, if $\phi(s) = (s, \phi^A(s))$, $s \in U$, then $j_t^1\phi$ has coordinates

$$\left(t, \phi^A(t), \frac{\mathrm{d}\phi^A}{\mathrm{d}s}(t) \right).$$

Therefore, if E has dimension $(n+1)$, $J^1\pi$ has dimension $(2n+1)$ and it is a fibred manifold over E and $\mathbb{R}$ with canonical projections $\pi_{1,0} : J^1\pi \longrightarrow E$ and $\pi_1 : J^1\pi \longrightarrow \mathbb{R}$, respectively. In local coordinates, we have

$$\pi_{1,0}(t, q^A, v^A) = (t, q^A) \qquad \pi_1(t, q^A, v^A) = t \qquad \pi(t, q^A) = t.$$

Jet manifolds $J^1\pi$ will be evolution spaces for time-dependent mechanics.

We define a canonical embedding $\iota : J^1\pi \longrightarrow TE$ as follows:

$$\iota(j_t^1\phi) = \dot{\phi}(t)$$

where $\dot{\phi}(t) \in T_{\phi(t)}E$ is the tangent vector at t of the curve $\phi(s)$. If we take local coordinates (t, q^A, τ, τ^A), we have

$$\iota(t, q^A, v^A) = (t, q^A, 1, v^A).$$

3. The vertical endomorphism

There exists a canonical endomorphism $\tilde{J}$ of $TJ^1\pi$, i.e. a tensor field of type (1, 1) on $J^1\pi$, defined as follows [30]. Let be $\tilde{X} \in T_{j_t^1\phi}(J^1\pi)$, and take its projections to E and $\mathbb{R}$:

$$T\pi_{1,0}(\tilde{X}) \in T_{\phi(t)}E \qquad T\pi_1(\tilde{X}) \in T_t\mathbb{R}.$$

Therefore, we have $T\pi_{1,0}(\tilde{X}) - T\phi(T\pi_1(\tilde{X})) \in (V\pi)_{\phi(t)}$, where $V\pi$ is the vertical subbundle of TE consisting of π-vertical tangent vectors on E. Now, we put

$$\tilde{J}(\tilde{X}) = (T\pi_{1,0}(\tilde{X}) - T\phi(T\pi_1(\tilde{X})))^{\mathrm{v}}_{/J^1\pi}$$

where the v means the vertical lift of a tangent vector at E to TE.

In local coordinates we obtain

$$\tilde{J}\left(\frac{\partial}{\partial t}\right) = -v^A \frac{\partial}{\partial v^A} \qquad \tilde{J}\left(\frac{\partial}{\partial q^A}\right) = \frac{\partial}{\partial v^A} \qquad \tilde{J}\left(\frac{\partial}{\partial v^A}\right) = 0$$

or, equivalently,

$$\tilde{J} = (\mathrm{d}q^A - v^A \mathrm{d}t) \otimes \frac{\partial}{\partial v^A}.$$

If we denote by $\theta^A = \mathrm{d}q^A - v^A\,\mathrm{d}t$ the set of local contact forms on $J^1\pi$, we obtain the more familiar expression

$$\tilde{J} = \theta^A \otimes \frac{\partial}{\partial v^A}.$$

4. Second-order differential equations

The manifold $J^2\pi$ of 2-jets of local sections is defined in a similar way:

$$J^2\pi = \left\{ \begin{array}{c} j_t^2\phi/\phi : U \subset \mathbb{R} \longrightarrow E, \pi \circ \phi = \mathrm{id}_U \\ U \text{ open neighbourhood of } t \end{array} \right\}.$$

We take local coordinates (t, q^A, v^A, a^A) on $J^2\pi$. $J^2\pi$ is a fibred manifold over $J^1\pi$, E and $\mathbb{R}$ with canonical projections

$$\pi_{2,1} : J^2\pi \longrightarrow J^1\pi \qquad \pi_{2,0} : J^2\pi \longrightarrow E \qquad \pi_2 : J^2\pi \longrightarrow \mathbb{R}$$

locally given by

$$\pi_{2,1}(t, q^A, v^A, a^A) = (t, q^A, v^A) \qquad \pi_{2,0}(t, q^A, v^A, a^A) = (t, q^A)$$
$$\pi_2(t, q^A, v^A, a^A) = t.$$

There exists a natural inclusion of $J^2\pi$ into the 1-jet manifold $J^1\pi_1$. In fact, define

$$\begin{aligned} j : J^2\pi &\hookrightarrow J^1\pi_1 \\ j_t^2\phi &\longmapsto j_t^1\psi \end{aligned}$$

where $\psi(s) = j_s^1\phi$. In local coordinates we obtain

$$j(t, q^A, v^A, a^A) = (t, q^A, v^A, v^A, a^A).$$

Moreover, there exists a natural embedding of $J^1\pi_1$ into $TJ^1\pi$. So, we have the following chain of embeddings:

$$J^2\pi \overset{j}{\hookrightarrow} J^1\pi_1 \overset{u}{\hookrightarrow} TJ^1\pi.$$

We will consider a special class of vector fields on $J^1\pi$.

Definition 4.1. We say that a vector field ξ on $J^1\pi$ is a non-autonomous second-order differential equation (NSODE for simplicity) if $\xi : J^1\pi \longrightarrow TJ^1\pi$ takes values into $(u \circ j)(J^2\pi)$.

Therefore, ξ is a NSODE iff it has the following local expression,

$$\xi(t, q^A, v^A) = \frac{\partial}{\partial t} + v^A \frac{\partial}{\partial q^A} + \xi^A \frac{\partial}{\partial v^A}$$

where $\xi^A = \xi^A(t, q^A, v^A)$.

If we put $\eta = (\pi_1)^*(\mathrm{d}t)$, we obtain the following geometrical characterization of a NSODE.

Proposition 4.2. ξ is a NSODE iff $\tilde{J}(\xi) = 0$ and $\eta(\xi) = 1$.

Notice that a local section ϕ of $\pi : E \longrightarrow \mathbb{R}$ may be viewed as a curve in E.

Definition 4.3. A local section ϕ of $\pi : E \longrightarrow \mathbb{R}$ is a solution of a NSODE ξ if the 1-jet prolongation $j^1\phi$ of ϕ to $J^1\pi$ is an integral curve of ξ.

Thus, $\phi(t) = (t, \phi^A(t))$ is a solution of ξ iff it satisfies the following system of non-autonomous differential equations of second order:

$$\frac{\mathrm{d}^2\phi^A}{\mathrm{d}t^2} = \xi^A\left(t, \phi^B, \frac{\mathrm{d}\phi^B}{\mathrm{d}t}\right) \qquad \frac{\mathrm{d}\phi^A}{\mathrm{d}t} = v^A.$$

It should be remarked that an integral curve σ of a NSODE ξ is necessarily a 1-jet prolongation, say $\sigma = j^1\phi$, where ϕ is a solution of ξ.

Remark 4.4. If E is the trivial fibration $pr_{\mathbb{R}} : E = \mathbb{R} \times Q \longrightarrow \mathbb{R}$, we have canonical identifications

$$J^1 pr_{\mathbb{R}} = \mathbb{R} \times TQ \qquad J^2 pr_{\mathbb{R}} = \mathbb{R} \times T^2Q \qquad J^1(pr_{\mathbb{R}})_1 = \mathbb{R} \times T(TQ)$$

where T^2Q is the tangent bundle of order 2 of Q.

5. Lagrangian mechanics in jet manifolds

Let $L : J^1\pi \longrightarrow \mathbb{R}$ be a non-autonomous or time-dependent Lagrangian function. Define the Poincaré–Cartan forms associated to L by

$$\Theta_L = L\eta + \tilde{J}^*(\mathrm{d}L) \text{ (Poincaré–Cartan 1-form)}$$
$$\Omega_L = -\mathrm{d}\Theta_L \text{ (Poincaré–Cartan 2-form).}$$

Denote by $\tilde{p}_A = \partial L/\partial v^A$ the generalized momenta. Then we have

$$\Theta_L = (L - v^A\tilde{p}_A)\,\mathrm{d}t + \tilde{p}_A\,\mathrm{d}q^A.$$

Of course, we also have

$$\Theta_L = L\,\mathrm{d}t + \tilde{p}_A\theta^A.$$

We say that L is regular if and only if the Hessian matrix

$$\left(\frac{\partial^2 L}{\partial v^A \partial v^B}\right)$$

is non-singular. So, L is regular iff (Ω_L, η) is a cosymplectic structure on $J^1\pi$. This means that Ω_L and η are closed and $\Omega_L^n \wedge \eta$ is a volume form (see [6, 13, 20]). In this case, there exists a unique vector field ξ_L on $J^1\pi$ such that

$$i_{\xi_L}\Omega_L = 0 \qquad i_{\xi_L}\eta = 1. \tag{1}$$

In other words, if $b_L : TJ^1\pi \longrightarrow T^*J^1\pi$ is the vector bundle isomorphism defined by $b_L(X) = i_X\Omega_L + \eta(X)\eta$, we have $\xi_L = b_L^{-1}(\eta)$. ξ_L is the Reeb vector field of the cosymplectic structure (Ω_L, η), and it will be called the Euler–Lagrange vector field.

Suppose that ξ_L is locally given by

$$\xi_L = \frac{\partial}{\partial t} + X^A\frac{\partial}{\partial q^A} + \xi^A\frac{\partial}{\partial v^A}.$$

A direct computation from (1) gives

$$v^A X^B \frac{\partial \tilde{p}_A}{\partial q^B} - X^B\frac{\partial L}{\partial q^B} + X^B\frac{\partial \tilde{p}_B}{\partial t} + \xi^B v^A \frac{\partial \tilde{p}_A}{\partial v^B} = 0 \tag{2}$$

$$-\frac{\partial \tilde{p}_A}{\partial t} - v^B\frac{\partial \tilde{p}_B}{\partial q^A} + \frac{\partial L}{\partial q^A} + X^B\frac{\partial \tilde{p}_B}{\partial q^A} - X^B\frac{\partial \tilde{p}_A}{\partial q^B} - \xi^B\frac{\partial \tilde{p}_A}{\partial v^B} = 0 \tag{3}$$

$$(X^B - v^B)\frac{\partial \tilde{p}_B}{\partial v^A} = 0. \tag{4}$$

From (4) and since L is regular, we deduce that $X^A = v^A$. Thus, (2) and (3) become

$$v^A\left[\frac{\partial \tilde{p}_A}{\partial t} + v^B\frac{\partial \tilde{p}_A}{\partial q^B} + \xi^B\frac{\partial \tilde{p}_A}{\partial v^B} - \frac{\partial L}{\partial q^A}\right] = 0 \tag{5}$$

$$\frac{\partial \tilde{p}_A}{\partial t} + v^B\frac{\partial \tilde{p}_A}{\partial q^B} + \xi^B\frac{\partial \tilde{p}_A}{\partial v^B} - \frac{\partial L}{\partial q^A} = 0. \tag{6}$$

Therefore, we have the following.

Theorem 5.1. (i) ξ_L is a NSODE.

(ii) The solutions of ξ_L are just the solutions of the Euler–Lagrange equations for L

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^A}\right) - \frac{\partial L}{\partial q^A} = 0 \qquad v^A = \frac{\mathrm{d}q^A}{\mathrm{d}t}. \tag{7}$$

6. Non-holonomic Lagrangian mechanics. Motion equations

Suppose that $L : J^1\pi \longrightarrow \mathbb{R}$ is a regular Lagrangian subjected to a set of non-holonomic constraints given by a m-codimensional distribution D on E. This means that the only allowable evolutions $j^1_t\phi$ have to belong to D. More precisely, the tangent vectors $\dot\phi(t) \in T_{\phi(t)}E$ have to be in $D_{\phi(t)}$. It should be noted that a compatibility condition on D has to be assumed. In fact, if D^0 is the annihilator of D, we will assume that $\pi^*(\mathrm{d}t)_x \notin (D^0)_x$, or, equivalently, $D^0 \wedge \pi^*(\mathrm{d}t) \neq 0$. Remark that if $\pi^*(\mathrm{d}t) \in D^0$, then $D \cap J^1\pi = \emptyset$ which implies the incompatibility of the constrained system.

Let μ_i be a local basis of D^0, i.e.

$$D^0 = \langle \mu_i \ / \ 1 \leqslant i \leqslant m \rangle.$$

We define two distributions D^{v} and D^{c} on $J^1\pi$ as follows. Let μ_i^{c} be the complete lift of μ_i to TE. Let us recall that if $\mu_i = (\mu_i)_A\, \mathrm{d}q^A + h_i\, \mathrm{d}t$, then

$$\begin{aligned}\mu_i^{\mathrm{c}} &= (\mu_i)_A^{\mathrm{c}}\, \mathrm{d}q^A + (\mu_i)_A^{\mathrm{v}}\, \mathrm{d}\tau^A + h_i^{\mathrm{c}}\mathrm{d}t + h_i \mathrm{d}\tau \\ &= \left(\tau \frac{\partial(\mu_i)_A}{\partial t} + \tau^B \frac{\partial(\mu_i)_A}{\partial q^B}\right)\mathrm{d}q^A + (\mu_i)_A\, \mathrm{d}\tau^A + \left(\tau \frac{\partial h_i}{\partial t} + \tau^B \frac{\partial h_i}{\partial q^B}\right)\mathrm{d}t + h_i\, \mathrm{d}\tau.\end{aligned}$$

Here μ_i^{v} denotes the vertical lift of μ_i to TE, i.e. the pull-back of μ_i by the canonical projection $\tau_E : TE \longrightarrow E$. Hence, its restriction to $J^1\pi$ is given by

$$\mu^{\mathrm{c}}_{i\,/J^1\pi} = \left(\frac{\partial(\mu_i)_A}{\partial t} + v^B \frac{\partial(\mu_i)_A}{\partial q^B}\right)\mathrm{d}q^A + (\mu_i)_A\, \mathrm{d}v^A + \left(\frac{\partial h_i}{\partial t} + v^B \frac{\partial h_i}{\partial q^B}\right)\mathrm{d}t.$$

We put $\bar\mu_i = \tilde J^*(\mu^{\mathrm{c}}_{i\,/J^1\pi})$. Thus, we get

$$\begin{aligned}\bar\mu_i &= (\mu_i)_A\, \mathrm{d}q^A - v^A(\mu_i)_A\, \mathrm{d}t \\ &= (\mu_i)_A\theta^A.\end{aligned}$$

Now, we define D^{v} and D^{c} by prescribing that their annihilators are locally generated by $\{\bar\mu_i\}$ and $\{\bar\mu_i, \mu^{\mathrm{c}}_{i\,/J^1\pi}\}$, i.e.

$$(D^{\mathrm{v}})^0 = \langle \bar\mu_i \rangle \qquad (D^{\mathrm{c}})^0 = \langle \bar\mu_i, \mu^{\mathrm{c}}_{i\,/J^1\pi} \rangle.$$

First of all, note that $\{\bar\mu_i, \mu^{\mathrm{c}}_{i\,/J^1\pi}\}$ are linearly independent at every point of $J^1\pi$. This follows taking into account that, from the assumption on D, the local 1-forms $\{(\mu_i)_A\, \mathrm{d}q^A\}$ are linearly independent. Secondly, $(D^{\mathrm{v}})^0$ and $(D^{\mathrm{c}})^0$ are well defined along $\tilde D = D \cap J^1\pi$. In fact, let $\{\mu_i'\}$ be another local basis of D^0. Thus, we have

$$\mu_i' = \Lambda_i^j \mu_j$$

where (Λ_i^j) is a non-singular matrix at every point in the overlapping of the two neighbourhoods where μ_i and μ_i' are defined. The following formulae are obtained by a direct computation

$$\begin{aligned}(\mu_i')^{\mathrm{c}}_{/J^1\pi} &= ((\Lambda_i^j)^{\mathrm{c}} \circ \iota)\pi^*_{1,0}(\mu_j) + \Lambda_i^j \mu^{\mathrm{c}}_{j\,/J^1\pi} \\ \bar\mu_i' &= \Lambda_i^j \bar\mu_j.\end{aligned} \tag{8}$$

From (8) it is easy to prove that D^{v} and D^{c} are well defined along $\tilde D = D \cap J^1\pi$.

Now, the constrained motion equations can be written as follows

$$i_X\Omega_L \in (D^{\mathrm{v}})^0 \qquad i_X\eta = 1 \qquad X \in D^{\mathrm{c}} \tag{9}$$

along the points of $\tilde D$.

In fact, (9) can be equivalently written as

$$i_X \Omega_L = \lambda^i \bar{\mu}_i \qquad i_X \,\mathrm{d}t = 1 \qquad \mu^{\mathrm{c}}_{i\,/J^1\pi}(X) = 0 \qquad \bar{\mu}_i(X) = 0 \tag{10}$$

where λ^i are some Lagrange multipliers to be determined [25].

Note that the first two equations in (9) imply that any solution X has to be a NSODE, and, then, the third equation in (9) becomes

$$\left(\frac{\partial h_i}{\partial t} + v^B \frac{\partial h_i}{\partial q^B}\right) + v^A \left(\frac{\partial (\mu_i)_A}{\partial t} + v^B \frac{\partial (\mu_i)_A}{\partial q^B}\right) + (\mu_i)_A X(v^A) = 0. \tag{11}$$

Now, let $\phi_i = (\hat{\mu}_i)_{/J^1\pi}$ be the restriction of the function $\hat{\mu}_i$ to $J^1\pi$. Let us recall that given a 1-form μ on a manifold N, we define an evaluation function $\hat{\mu}$ on TN by $\hat{\mu}(X) = \langle \mu, X\rangle$. Since $\hat{\mu}_i(t, q^A, \tau, \tau^A) = (\mu_i)_A \tau^A + h_i \tau$, we deduce that

$$\phi_i(t, q^A, v^A) = (\mu_i)_A v^A + h_i \tag{12}$$

which is the usual form of the constraints in the local analysis (see [25]). By comparing (11) and (12) we deduce that the condition $X \in D^{\mathrm{c}}$ is equivalent to ask that X has to be tangent to the submanifold of $J^1\pi$ locally defined by the vanishing of the ϕ_i's. This submanifold is just $\tilde{D} = D \cap J^1\pi$, where D is now considered as a submanifold of TE. Note that the functions ϕ_i are independent since $D^0 \wedge \pi^*(\mathrm{d}t) \neq 0$.

From (10) we deduce that the solutions of X satisfy the following system of second-order differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^A}\right) - \frac{\partial L}{\partial q^A} = -\lambda^i (\mu_i)_A \qquad v^A = \frac{\mathrm{d}q^A}{\mathrm{d}t}$$

subjected to the constraints $\phi_i(t, q^A, v^A) = 0$.

Remark 6.1. Notice that equations (9) are restricted to the submanifold $\tilde{D}$, since $\bar{\mu}_i(X) = \phi_i$=0.

7. Solving the motion equations

In this section we shall give a procedure to solve equations (9) by using a very geometrical method. First of all, we give the following definition.

Let S be the distribution on $J^1\pi$ obtained from $(D^{\mathrm{v}})^0$ by means of the isomorphism $b_L : T(J^1\pi) \longrightarrow T^*(J^1\pi)$, namely

$$S(x) = b_L(x)^{-1}\left((D^{\mathrm{v}})^0_x\right) \qquad \forall x \in \tilde{D}.$$

In fact, S is a distribution along the points of $\tilde{D}$. If we put

$$i_{Z_i}\Omega_L + \eta(Z_i)\eta = \bar{\mu}_i$$

then S is locally generated by the Z_i's. Notice that Z_i is completely characterized by the conditions

$$i_{Z_i}\Omega_L = \bar{\mu}_i \qquad i_{Z_i}\eta = 0.$$

Thus, Z_i is a $\pi_{1,0}$-vertical vector field along $\tilde{D}$.

Definition 7.1. The constrained system is said to be regular if

$$S_x \cap T_x\tilde{D} = 0 \qquad \forall x \in \tilde{D}.$$

Now, let us explain the meaning of the regularity condition.

Put $\mathcal{C}_{ij} = Z_i(\phi_j)$ and take the matrix $\mathcal{C} = (\mathcal{C}_{ij})$. Then, we have

Proposition 7.2. The constrained system is regular iff the matrices $\mathcal{C} = (\mathcal{C}_{ij})$ are non-singular.

Proof. Suppose that the constrained system is regular. Take an arbitrary linear combination of columns of $\mathcal{C}$ at some point x such that

$$\sum_{i=1}^{m} \lambda^i Z_i(x)(\phi_j) = 0.$$

Thus, $\sum \lambda^i Z_i(x) \in T_x\tilde{D}$ which implies that $\sum \lambda^i Z_i(x) = 0$, and hence $\lambda^1 = \lambda^2 = \cdots = \lambda^m = 0$.

Conversely, suppose $\mathcal{C}$ be non-singular and let be $X \in S_x \cap T_x\tilde{D}$. Thus, $X = \sum \lambda^i Z_i(x)$ and $X(\phi_j) = 0$, $\forall j, 1 \leqslant j \leqslant m$ which implies that $\sum \lambda^i Z_i(\phi_j) = 0$. Therefore, we deduce that $\lambda^1 = \cdots = \lambda^m = 0$, and consequently $X = 0$. □

Proposition 7.3. If the Hessian matrix

$$\left(\frac{\partial^2 L}{\partial v^A \partial v^B}\right)$$

is positive or negative definite at each point $x \in \tilde{D}$, then the constrained system is regular.

Proof. The result follows since

$$\mathcal{C}_{ij} = -W^{AB}(\mu_i)_A(\mu_j)_B$$

where (W^{AB}) denotes the inverse matrix of the Hessian matrix $(\partial^2 L/\partial v^A \partial v^B)$. □

Remark 7.4. The last proposition clarifies the usual assumption on the positive or negative character of the Hessian matrix of L. It is nothing but a sufficient condition to ensure the regularity of the constrained system. For instance, let g be a Riemannian metric on the vertical bundle $V\pi$ such that $g = g_{AB}(t, q)\,\mathrm{d}q^A\,\mathrm{d}q^B$. As we know, $\pi_{1,0} : J^1\pi \longrightarrow E$ is an affine bundle modelled on the vertical vector bundle $V\pi \longrightarrow E$. The choice of a global section s of $\pi_{1,0}$ (which is equivalent to the choice of a connection in the fibration $\pi : E \longrightarrow \mathbb{R}$ [30]) leads us to define an associated kinetic energy by $L(t, q^A, v^A) = g_{AB}v^Av^B + 2g_{AB}v^As^B + g_{AB}s^As^B$, where $s(t, q^A) = (t, q^A, s^A(t, q))$. Therefore, the Hessian matrix becomes $(\partial^2 L/\partial v^A \partial v^B = g_{AB})$. In case of E be the trivial fibration $pr_{\mathbb{R}} : E = \mathbb{R} \times Q \longrightarrow \mathbb{R}$, we can take the standard connection such that $s(t, q^A) = (t, q^A, 0)$. Thus, the associated Lagrangian function is just $L(t, q^A, v^A) = g_{AB}v^Av^B$.

Since $\dim \tilde{D} = 2n + 1 - m$ and $\dim S(x) = m$, $\forall x \in \tilde{D}$, we conclude the following.

Proposition 7.5. If the constrained system is regular, we have

$$T_x(J^1\pi) = S_x \oplus T_x\tilde{D} \qquad \forall x \in \tilde{D}. \tag{13}$$

Moreover, we can realize this splitting as follows. Define a linear map

$$\mathcal{Q}_x : T_x(J^1\pi) \longrightarrow T_x(J^1\pi)$$

for every $x \in \tilde{D}$, by putting

$$\mathcal{Q}_x = \mathcal{C}^{ij}(x) Z_j(x) \otimes \mathrm{d}\phi_i(x).$$

A direct computation shows that $\mathcal{Q}_x^2 = \mathcal{Q}_x$ and $\mathcal{Q}_x(X) \in S(x)$, for all $x \in \tilde{D}$ and for all $X \in T_x(J^1\pi)$. Thus,

$$X = \mathcal{Q}_x(X) + (X - \mathcal{Q}_x(X))$$

is the splitting given in (13).

The above splitting is intrinsic. Nevertheless, in order to clarify our procedure, we shall study the behaviour of $\mathcal{Q}$ by a change of local basis. Take another local basis $\{\mu_i'\}$ of D^0 such that

$$\mu_i' = \Lambda_i^j \mu_j.$$

Hence, we obtain

$$(\mu_i')_A = \Lambda_i^j (\mu_j)_A \qquad h_i' = \Lambda_i^j h_j$$

where $\mu_i' = (\mu_i')_A \, \mathrm{d}q^A + h_i' \, \mathrm{d}t$. Therefore, the new constraint functions defining $\tilde{D}$ are

$$\phi_i' = \Lambda_i^j \phi_j. \tag{14}$$

On the other hand, we get

$$Z_i' = \Lambda_i^j Z_j \tag{15}$$

where $\{Z_i'\}$ is the new local basis of S. From (14) and (15) we have

$$\begin{aligned}
\mathcal{C}_{ij}' &= Z_i'(\phi_j') = \Lambda_i^r Z_r(\Lambda_j^s \phi_s) \\
&= \Lambda_i^r \Lambda_j^s Z_r(\phi_s) + \Lambda_i^r \phi_s Z_r(\Lambda_j^s) \\
&= \mathcal{C}_{rs} \Lambda_i^r \Lambda_j^s
\end{aligned}$$

along the points of $\tilde{D}$. Thus,

$$(\mathcal{C}')^{ij} = \mathcal{C}^{rs} (\Lambda^{-1})_r^i (\Lambda^{-1})_s^j$$

along $\tilde{D}$. This implies

$$\begin{aligned}
\mathcal{Q}' &= (\mathcal{C}')^{ij} Z_j' \otimes \mathrm{d}\phi_i' \\
&= \mathcal{C}^{rs} (\Lambda^{-1})_r^i (\Lambda^{-1})_s^j \Lambda_j^a Z_a \otimes \mathrm{d}(\Lambda_i^b \phi_b) \\
&= \mathcal{Q} + \mathcal{C}^{ra} (\Lambda^{-1})_r^i \phi_b Z_a \otimes \mathrm{d}\Lambda_i^b \\
&= \mathcal{Q}
\end{aligned}$$

along $\tilde{D}$. Therefore, $\mathcal{Q}$ is well defined along $\tilde{D}$ and it is a tensor field of type $(1, 1)$ on $J^1\pi$ along $\tilde{D}$. Since $\mathcal{Q}^2 = \mathcal{Q}$, we have obtained an almost product structure on $J^1\pi$ along $\tilde{D}$.

If $\mathcal{P} = \mathrm{id} - \mathcal{Q}$, then $\mathcal{P}(\xi_L)(x) \in T_x\tilde{D}$, $\forall x \in \tilde{D}$. Thus, $\mathcal{P}(\xi_{L/\tilde{D}})$ is tangent to $\tilde{D}$, say $\mathcal{P}(\xi_{L/\tilde{D}}) \in \mathfrak{X}(\tilde{D})$. Moreover,

$$\begin{aligned}
\mathcal{P}(\xi_{L/\tilde{D}}) &= \xi_{L/\tilde{D}} - \mathcal{Q}(\xi_{L/\tilde{D}}) \\
&= \xi_{L/\tilde{D}} - \mathcal{C}^{ij} \xi_{L/\tilde{D}}(\phi^i) Z_j
\end{aligned}$$

which implies that $\mathcal{P}(\xi_{L/\tilde{D}})$ is a solution of (9). So, we have proved the following.

Proposition 7.6. If the constrained system is regular, there exists an almost product structure $(\mathcal{P}, \mathcal{Q})$ along the constraint submanifold $\tilde{D} = D \cap J^1\pi$ such that $\mathcal{P}(\xi_{L/\tilde{D}})$ is tangent to $\tilde{D}$, and is a solution of the constrained dynamics.

Remark 7.7. Since (Ω_L, η) is cosymplectic, $\mathcal{P}(\xi_{L/\tilde{D}})$ is in fact the only solution of the constrained motion equations.

From the regularity of the local matrices $\mathcal{C}$ we deduce that $(\mathcal{P}, \mathcal{Q})$ may be extended (in many ways) to an open neighbourhood of $\tilde{D}$. Therefore, ξ may also be extended to an open neighbourhood of $\tilde{D}$. This fact will be used in the following lemmas.

Lemma 7.8. Given a regular constrained system (L, D), the vector field ξ solving the constrained dynamics satisfies

$$\mathcal{L}_\xi \Theta_L = \mathrm{d}L - \mathcal{L}_{\mathcal{Q}(\xi_L)}\Theta_L$$

along the points of $\tilde{D}$, where $\mathcal{L}$ denotes the Lie derivative.

Proof. It follows since $\xi = \mathcal{P}(\xi_L) = \xi_L - \mathcal{Q}(\xi_L)$ and $\mathcal{L}_{\xi_L}\Theta_L = \mathrm{d}L$. □

Lemma 7.9. Under the same hypothesis as in lemma 7.8, we have

$$\mathcal{L}_{\mathcal{Q}(\xi_L)}\Theta_L \in (D^{\mathrm{v}})^0.$$

Proof. Since $\mathcal{Q}(\xi_L) = \sum_{j=1}^m \Lambda^j Z_j$, with $\Lambda^j = \mathcal{C}^{ij}\xi_L(\phi_i)$, we deduce that

$$\begin{aligned}\mathcal{L}_{\mathcal{Q}(\xi_L)}\Theta_L &= \mathcal{L}_{\sum_{j=1}^m \Lambda^j Z_j}\Theta_L \\ &= i_{\sum_{j=1}^m \Lambda^j Z_j} d\Theta_L + \mathrm{d}(i_{\sum_{j=1}^m \Lambda^j Z_j}\Theta_L) \\ &= -i_{\sum_{j=1}^m \Lambda^j Z_j}\Omega_L = -\sum_{j=1}^m \Lambda^j \bar{\mu}_j\end{aligned}$$

since the vector fields Z_j are $\pi_{1,0}$-vertical and Θ_L is semibasic. □

8. The constrained Poincaré–Cartan 2-form

Let $L : J^1\pi \longrightarrow \mathbb{R}$ be a regular constrained system subjected to a set of non-holonomic constraints given by a m-codimensional distribution D on E. For every point $x \in \tilde{D} = D \cap J^1\pi$, define

$$\omega(x) = \Omega_L(x) - (i_{\mathcal{Q}(\xi_L)(x)}\Omega_L(x)) \wedge \eta(x).$$

Hence ω is a 2-form on $J^1\pi$ along $\tilde{D}$. We also have that $\eta(x) \wedge \omega^n(x) \neq 0$ for all $x \in \tilde{D}$. Thus, there exists a unique vector field X on $J^1\pi$ along $\tilde{D}$ such that

$$i_X\omega = 0 \qquad i_X\eta = 1. \tag{16}$$

In fact, a direct computation proves that $X = \mathcal{P}(\xi_{L/\tilde{D}})$.

Next, we get the following.

Theorem 8.1. If $\tilde{\omega}$ and $\tilde{\eta}$ are the restrictions of ω and η to the constrained submanifold $\tilde{D} = D \cap J^1\pi$ then the solution $\mathcal{P}(\xi_{L/\tilde{D}})$ of the constrained dynamics verifies the equations

$$i_X\tilde{\omega} = 0 \qquad i_X\tilde{\eta} = 1. \tag{17}$$

Moreover, the unique NSODE X on $\tilde{D}$ satisfying (17) is just $\mathcal{P}(\xi_{L/\tilde{D}})$.

Proof. Since the vector field $\mathcal{P}(\xi_{L/\tilde{D}})$ satisfies (16), then it also verifies (17).

Now, let X be a NSODE on $\tilde{D}$ (that is, $\tilde{J}X = 0$) such that $i_X\tilde{\omega} = 0$ and $i_X\tilde{\eta} = 1$. Then, we have that

$$(i_X\omega)(\mathcal{P}(Y)) = 0 \tag{18}$$

for all vector fields Y on $J^1\pi$ along $\tilde{D}$.

On the other hand, if Z is a vector field on $J^1\pi$ along $\tilde{D}$, using that $\mathcal{Q}(Z)$ is $\pi_{1,0}$-vertical and the fact that X is a NSODE, we obtain

$$(i_X\omega)(\mathcal{Q}(Z)) = -(i_{\mathcal{Q}(Z)}\Omega_L)(X) - (i_{\mathcal{Q}(\xi_L)}\Omega_L)(X)\eta(\mathcal{Q}(Z)) + (i_{\mathcal{Q}(\xi_L)}\Omega_L)(\mathcal{Q}(Z)) = 0. \tag{19}$$

Finally, from (18) and (19), we conclude that $i_X\omega = 0$ which implies that $X = \mathcal{P}(\xi_{L/\tilde{D}})$. □

Definition 8.2. The 2-form $\tilde{\omega}$ is said to be the constrained Poincaré–Cartan 2-form.

Remark 8.3. (i) The 2-form $\tilde{\omega}$ coincides (up to the sign) with the one obtained by Saunders *et al* (see [31]). It should be remarked that our result holds for arbitrary regular non-holonomic Lagrangian systems, without any assumption on the positive or negative definiteness of L.

(ii) Note that $(\tilde{\omega}, \tilde{\eta})$ is no longer cosymplectic so that it may be another solution of the equations

$$i_X\tilde{\omega} = 0 \qquad i_X\tilde{\eta} = 1.$$

Example 8.4. *(The curve of pursuit.)* Suppose that a point A moves on the axis Ox, the distance OA being a prescribed function $f(t)$ of t. The particle of mass m, whose position at time t is (x, y), moves in the xy-plane, and is constrained so that at each instant its velocity is directed towards A. This curve is called *curve of pursuit* (see [24]).

Consider the trivial bundle $\pi : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$, $\pi(t, x, y) = t$ and the jet bundle $J^1\pi$ with coordinates $(t, x, y, \dot{x}, \dot{y})$. We can describe this system by the Lagrangian $L : J^1\pi \longrightarrow \mathbb{R}$

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

and the distribution D globally annihilated by the 1-form

$$\mu = y\,\mathrm{d}x + (f(t) - x)\,\mathrm{d}y.$$

A direct computation shows that

$$\Theta_L = -\tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2)\,\mathrm{d}t + m\dot{x}\,\mathrm{d}x + m\dot{y}\,\mathrm{d}y$$
$$\Omega_L = m\dot{x}\,\mathrm{d}\dot{x} \wedge \mathrm{d}t + m\dot{y}\,\mathrm{d}\dot{y} \wedge \mathrm{d}t + m\,\mathrm{d}x \wedge \mathrm{d}\dot{x} + m\,\mathrm{d}y \wedge \mathrm{d}\dot{y}$$
$$\xi_L = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y}.$$

Therefore, the distribution D^{v} is defined by prescribing its annihilator be generated by the global 1-form

$$\bar{\mu} = y\,\mathrm{d}x + (f(t) - x)\,\mathrm{d}y - \dot{x}y\,\mathrm{d}t - \dot{y}(f(t) - x)\,\mathrm{d}t.$$

Hence, the distribution S is generated by the vector field

$$Z = -\frac{y}{m}\frac{\partial}{\partial \dot{x}} - \frac{(f(t) - x)}{m}\frac{\partial}{\partial \dot{y}}.$$

Since $S_x \cap T_x\tilde{D} = 0$ for all $x \in \tilde{D}$, we deduce that the constrained system is regular. Notice that L is the kinetic energy associated with the Riemannian metric g on $\mathbb{R}^2$ given by $g = m(\mathrm{d}x^2 + \mathrm{d}y^2)$ (see proposition 7.3 and remark 7.4).

From the decomposition $T_xJ^1\pi = S_x \oplus T_x\tilde{D}$, we get the complementary projectors

$$\mathcal{Q} = \mathcal{C}^{-1}Z \otimes \mathrm{d}\phi = \frac{1}{y^2 + (f(t)-x)^2}\left(y\frac{\partial}{\partial \dot{x}} + (f(t)-x)\frac{\partial}{\partial \dot{y}}\right)$$
$$\otimes(\dot{x}\,\mathrm{d}y + y\,\mathrm{d}\dot{x} + \frac{\partial f}{\partial t}\dot{y}\,\mathrm{d}t - \dot{y}\,\mathrm{d}x + (f(t)-x)\,\mathrm{d}\dot{y})$$
$$\mathcal{P} = \mathrm{id} - \mathcal{Q}$$

where

$$\mathcal{C} = -\frac{1}{m}(y^2 + (f(t)-x)^2) \qquad \phi = y\dot{x} + (f(t)-x)\dot{y}.$$

The solution of the constrained dynamics is the vector field

$$\xi = \mathcal{P}(\xi_{L/\tilde{D}}) = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} - \frac{y\dot{y}}{y^2 + (f(t)-x)^2}\left(\frac{\partial f}{\partial t}\right)\frac{\partial}{\partial \dot{x}}$$
$$-\frac{(f(t)-x)\dot{y}}{y^2 + (f(t)-x)^2}\left(\frac{\partial f}{\partial t}\right)\frac{\partial}{\partial \dot{y}}.$$

So, the solutions of the constrained motion equations are the solutions of the following system of non-autonomous second-order differential equations:

$$\ddot{x} = -\frac{y\dot{y}}{y^2 + (f(t)-x)^2}\frac{\partial f}{\partial t} \qquad \ddot{y} = -\frac{(f(t)-x)\dot{y}}{y^2 + (f(t)-x)^2}\frac{\partial f}{\partial t}$$

Finally, the constrained Poincaré–Cartan 2-form $\tilde{\omega}$ is the restriction to the constraint submanifold $\tilde{D}$ of the 2-form

$$\omega = m\dot{x}\,\mathrm{d}\dot{x} \wedge \mathrm{d}t + m\dot{y}\,\mathrm{d}\dot{y} \wedge \mathrm{d}t + m\,\mathrm{d}x \wedge \mathrm{d}\dot{x} + m\,\mathrm{d}y \wedge \mathrm{d}\dot{y}$$
$$+\frac{m\dot{y}}{y^2 + (f(t)-x)^2}\frac{\partial f}{\partial t}(y\,\mathrm{d}x \wedge \mathrm{d}t + (f(t)-x)\,\mathrm{d}y \wedge \mathrm{d}t).$$

Example 8.5. (*An special Čaplygin sleigh* [23], (p 94), [26, 31].) Let us consider the free motion of a solid body on a horizontal plane in the case when the projection of the centre of mass coincides with the point of contact of a sharp wheel and the plane.

Consider the trivial bundle $\pi : \mathbb{R} \times \mathbb{R}^2 \times S^1 \longrightarrow \mathbb{R}$, $\pi(t, x, y, \phi) = t$ and the jet bundle $J^1\pi$ with coordinates $(t, x, y, \phi, \dot{x}, \dot{y}, \dot{\phi})$. We can describe this system by the regular Lagrangian function $L : J^1\pi \longrightarrow \mathbb{R}$,

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{\phi}^2),$$

and the distribution D globally annihilated by the 1-form

$$\mu = \cos\phi\,\mathrm{d}y - \sin\phi\,\mathrm{d}x.$$

So, the constraints are given by $\psi(t, x, y, \phi, \dot{x}, \dot{y}, \dot{\phi}) = (\cos\phi)\dot{y} - (\sin\phi)\dot{x} = 0$. In an open set where $\tan\phi$ is defined, the constraints are given by $\dot{y} = \dot{x}\tan\phi$.

A direct computation shows that

$$\Theta_L = \dot{x}\,\mathrm{d}x + \dot{y}\,\mathrm{d}y + \dot{\phi}\,\mathrm{d}\phi - \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{\phi}^2)\,\mathrm{d}t$$
$$\Omega_L = \mathrm{d}x \wedge \mathrm{d}\dot{x} + \mathrm{d}y \wedge \mathrm{d}\dot{y} + \mathrm{d}\phi \wedge \mathrm{d}\dot{\phi} - \mathrm{d}t \wedge (\dot{x}\,\mathrm{d}\dot{x} + \dot{y}\,\mathrm{d}\dot{y} + \dot{\phi}\,\mathrm{d}\dot{\phi})$$
$$\xi_L = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{\phi}\frac{\partial}{\partial \phi}.$$

Because

$$\bar{\mu} = \cos\phi\,\mathrm{d}y - \sin\phi\,\mathrm{d}x + (\dot{x}\sin\phi - \dot{y}\cos\phi)\,\mathrm{d}t$$

the distribution S is generated by the vector field

$$Z = \sin\phi\frac{\partial}{\partial\dot{x}} - \cos\phi\frac{\partial}{\partial\dot{y}}.$$

Since $S_x \cap T_x\tilde{D} = 0$ for all $x \in \tilde{D}$, we deduce that the constrained system is regular. In fact, L is the kinetic energy associated with the Riemannian metric $g = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}\phi^2$ on $\mathbb{R}^2 \times S^1$ (see proposition 7.3 and remark 7.4).

The matrix $\mathcal{C}$ is just a real function, say $\mathcal{C} = Z(\psi) = -1$, and we get complementary projectors

$$\mathcal{Q} = -Z \otimes \mathrm{d}\psi \qquad \mathcal{P} = \mathrm{id} + Z \otimes \mathrm{d}\psi.$$

Finally, the solution of the constrained dynamics is the vector field

$$\xi = \mathcal{P}((\xi_L)_{/\tilde{D}}) = \left(\frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{\phi}\frac{\partial}{\partial\phi} - \dot{y}\dot{\phi}\frac{\partial}{\partial\dot{x}} + \dot{x}\dot{\phi}\frac{\partial}{\partial\dot{y}}\right)_{/\tilde{D}}.$$

However, along an open set U of $\tilde{D}$ for which $\cos\phi \neq 0$, we can choose local coordinates $(t, x, y, \phi, \dot{x}, \dot{\phi})$ so that ξ becomes

$$\xi = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + (\dot{x}\tan\phi)\frac{\partial}{\partial y} + \dot{\phi}\frac{\partial}{\partial\phi} - (\dot{x}\dot{\phi}\tan\phi)\frac{\partial}{\partial\dot{x}}.$$

Again by a straightforward computation we deduce that the constrained Poincaré–Cartan 2-form is given by

$$\omega = \mathrm{d}x \wedge \mathrm{d}\dot{x} + \mathrm{d}y \wedge \mathrm{d}\dot{y} + \mathrm{d}\phi \wedge \mathrm{d}\dot{\phi} - \mathrm{d}t \wedge (\dot{x}\,\mathrm{d}\dot{x} + \dot{y}\,\mathrm{d}\dot{y} + \dot{\phi}\,\mathrm{d}\dot{\phi} - \dot{y}\dot{\phi}\,\mathrm{d}x + \dot{x}\dot{\phi}\,\mathrm{d}y).$$

Thus, its restriction to U becomes

$$\tilde{\omega} = -((\mathrm{d}\Theta_L)_{/U} - \dot{x}\dot{\phi}\,\mathrm{d}t \wedge (\mathrm{d}y - \tan\phi\,\mathrm{d}x))$$

which is (up to the sign) the 2-form obtained in [31].

9. The singular case

Suppose now that the constrained system is not regular, that is, we have $S_x \cap T_x\tilde{D} \neq 0$, for some $x \in \tilde{D}$. From proposition 7.2, this fact is equivalent to the non-regularity of the local matrices $\mathcal{C} = (\mathcal{C}_{ij})$.

We consider the distribution S_L on $J^1\pi$ along the points of $\tilde{D}$ given by

$$(S_L)_x = S_x \oplus \langle \xi_L(x) \rangle$$

for all points $x \in \tilde{D}$.

We have

$$S_x \cap T_x\tilde{D} \subset (S_L)_x \cap T_x\tilde{D}$$

for any point $x \in \tilde{D}$.

In section 7, we have constructed an almost product structure $(\mathcal{P}, \mathcal{Q})$ on $J^1\pi$ along $\tilde{D}$ so that the unique solution ξ of the dynamics is just the projection by $\mathcal{P}$ of the Euler–Lagrange vector field ξ_L, that is, $\xi = \mathcal{P}((\xi_L)_{/\tilde{D}})$. In the regular case, we have that $\dim (S_L)_x \cap T_x\tilde{D} = 1$, and a generator of this vector space is precisely $\xi(x)$.

Now, consider the following subset in $\tilde{D}$:

$$\tilde{D}_2 = \{x \in \tilde{D}/S_x \cap T_x\tilde{D} \subsetneq (S_L)_x \cap T_x\tilde{D}\}$$

which is supposed to be a submanifold. At the points in $\tilde{D}_2$ there exists at least a tangent vector $X = \xi_L(x) + \lambda^i Z_i(x)$, for some real numbers $\lambda^i \in \mathbb{R}$, such that it belongs to $T_x\tilde{D}$. However, X is not necessarily tangent to $\tilde{D}_2$, and, therefore, we are compelled to define the submanifold $\tilde{D}_3$ of $\tilde{D}_2$ as follows:

$$\tilde{D}_3 = \{x \in \tilde{D}_2/S_x \cap T_x\tilde{D}_2 \subsetneq (S_L)_x \cap T_x\tilde{D}_2\}.$$

Proceeding further, we obtain the following sequence of constraint submanifolds

$$\cdots \to \tilde{D}_k \to \cdots \tilde{D}_3 \to \tilde{D}_2 \to \tilde{D}_1 = \tilde{D}$$

where, for any $k > 1$ we have

$$\tilde{D}_k = \{x \in \tilde{D}_{k-1}/S_x \cap T_x\tilde{D}_{k-1} \subsetneq (S_L)_x \cap T_x\tilde{D}_{k-1}\}.$$

In the following, we will suppose that this algorithm stabilizes, that is, there exists an integer $k \geqslant 1$ such that $\tilde{D}_{k+1} = \tilde{D}_k$ and $\dim \tilde{D}_k > 0$. We denote by $\tilde{D}_f = \tilde{D}_k$ the final constraint submanifold, and then there exists at least a vector field ξ on $\tilde{D}_f$ satisfying

$$(i_\xi\Omega_L \in (D^{\mathrm{v}})^0)_{/\tilde{D}_f} \qquad (i_\xi\eta = 1)_{/\tilde{D}_f}. \tag{20}$$

Along the points of $\tilde{D}_f$ we have the following strict inclusion

$$S_x \cap T_x\tilde{D}_f \subsetneq (S_L)_x \cap T_x\tilde{D}_f$$

for any point $x \in \tilde{D}_f$.

Then, there exist vector fields X on $\tilde{D}_f$ such that $X(x) \in (S_L)_x \cap T_x\tilde{D}_f$ but $X(x) \notin S_x \cap T_x\tilde{D}_f$. Therefore, we can select a vector field Y on $\tilde{D}_f$ such that $Y = (\xi_L + \lambda^i Z_i)_{/\tilde{D}_f}$ for some suitable values of the Lagrange multipliers λ^i on $\tilde{D}_f$. In particular we have shown that $\xi_L(x) \in S_x + T_x\tilde{D}_f$.

As in the regular case, it is possible to construct almost product structures along the points of $\tilde{D}_f$ such that the projection of the Euler–Lagrange vector field ξ_L gives us a solution of the constrained dynamics.

First of all, we will assume that the subspace $S_x \cap T_x\tilde{D}_f$ has constant dimension r for any point $x \in \tilde{D}_f$. Now, we split S_x as direct sum of two complementary subspaces, say

$$S_x = \check{S}_x \oplus (S_x \cap T_x\tilde{D}_f).$$

It is clear that $\dim \check{S}_x = m - r$, and this splitting is not unique.

Next, using that $T_x\tilde{D}_f \cap \check{S}_x = \{0\}$, we split the whole tangent space $T_x(J^1\pi)$:

$$T_x(J^1\pi) = \check{S}_x \oplus T_x\tilde{D}_f \oplus M_x, x \in \tilde{D}_f$$

where M_x is a suitable complementary subspace.

There exist three projectors associated with the above splitting:

$$\mathcal{Q}_x : T_x(J^1\pi) \longrightarrow \check{S}_x$$
$$(\mathcal{P}_1)_x : T_x(J^1\pi) \longrightarrow T_x\tilde{D}_f$$
$$(\mathcal{P}_2)_x : T_x(J^1\pi) \longrightarrow M_x.$$

Define the projector $\mathcal{P}_x = (\mathcal{P}_1)_x + (\mathcal{P}_2)_x$. Since $\xi_L(x) \in S_x + T_x\tilde{D}_f$, we deduce that $\mathcal{P}_x(\xi_L(x)) = (\mathcal{P}_1)_x(\xi_L(x))$, and along the points of $\tilde{D}_f$ we have that

$$\begin{aligned} i_{\mathcal{P}_x(\xi_L(x))}\Omega_L(x) &= i_{(\xi_L(x)-\mathcal{Q}_x(\xi_L(x)))}\Omega_L(x) \\ &= -i_{\mathcal{Q}_x(\xi_L(x))}\Omega_L(x) \in (D^{\mathrm{v}})^0_x \end{aligned}$$

and

$$
\begin{aligned}
i_{\mathcal{P}_x(\xi_L(x))}\eta(x) &= i_{(\xi_L(x)-\mathcal{Q}_x(\xi_L(x)))}\eta(x) \\
&= i_{\xi_L(x)}\eta(x) = 1.
\end{aligned}
$$

Moreover, $\mathcal{P}_x(\xi_L(x)) \in T_x\tilde{D}_f$. We deduce that $\mathcal{P}(\xi_{L/\tilde{D}_f})$ is a solution of the constrained dynamics and there exists an ambiguity of the solution of the dynamics because any vector field of the form $\mathcal{P}(\xi_{L/\tilde{D}_f}) + X$, with $X \in S \cap T\tilde{D}_f$ is a solution of the dynamics, too.

We have chosen complementary distributions $\check{S}$ and M in order to obtain the dynamics. Note that it is possible to realize both decompositions, say $S = \check{S} \oplus (T\tilde{D}_f \cap S)$ and $T(J^1\pi) = \check{S} \oplus T\tilde{D}_f \oplus M$ along $\tilde{D}_f$. In fact suppose that $\tilde{D}_f$ is locally defined by the vanishing of the functions Ψ_i, $i = 1, \ldots, p$, and take a local basis $\{\mu_i\}$ of D^0. Denote by Z_i the corresponding $\pi_{1,0}$-vertical vector fields along $\tilde{D}$ and by $\mathcal{C}_f$ the matrix with entries $(\mathcal{C}_f)_{ij} = Z_i(\Psi_j)$. We consider the linear map

$$
\Psi_x : S_x \longrightarrow \mathbb{R}^p \ , u \in S_x \longmapsto (u(\Psi_1), \ldots, u(\Psi_p))
$$

for a point $x \in \tilde{D}_f$. It is evident that $\ker\Psi_x = S_x \cap T_x\tilde{D}_f$. Furthermore, the associated matrix with Ψ_x with respect to the basis $\{Z_1(x), \ldots, Z_m(x)\}$ and the canonical basis of $\mathbb{R}^p$ is precisely $(\mathcal{C}_f)_x$.

We have assumed that $T\tilde{D}_f \cap S$ has constant rank r. Thus, the matrix $\mathcal{C}_f$ has constant rank $m - r$. Suppose that the matrix $\mathcal{C}'_f = ((\mathcal{C}_f)_{i'j'})$, $1 \leqslant i', j' \leqslant m - r$, is regular. In such a case, we define a projector $\mathcal{Q}$ by

$$
\mathcal{Q} = (\mathcal{C}'_f)^{i'j'} Z_{j'} \otimes \mathrm{d}\Psi_{i'}
$$

where $((\mathcal{C}'_f)^{i'j'})$ is the inverse matrix of $\mathcal{C}'_f$. Note that $\check{S} = \langle Z_{i'} \ / \ 1 \leqslant i' \leqslant m - r\rangle$. If we put $\mathcal{P} = \mathrm{id} - \mathcal{Q}$ we obtain an almost product structure $(\mathcal{P}, \mathcal{Q})$ along $\tilde{D}_f$. The decomposition $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$ is obtained by choosing a complementary M of $\check{S} \oplus TD_f$. This choice corresponds to the ambiguity in the determination of the remainder Lagrange multipliers. Indeed, if we compute $\mathcal{P}(\xi_{L/\tilde{D}_f})$ we obtain

$$
\mathcal{P}(\xi_{L/\tilde{D}_f}) = \Big(\xi_L - (\mathcal{C}'_f)^{i'j'}\xi_L(\Psi_{i'})Z_{j'}\Big)_{/\tilde{D}_f}
$$

and a general solution is of the form

$$
\mathcal{P}(\xi_{L/\tilde{D}_f}) + Y
$$

where $Y \in T\tilde{D}_f \cap S$. So, the only Lagrange multipliers determined are just the components of $Z_{j'}$'s.

10. The Hamiltonian formalism

Let $L : J^1\pi \to \mathbb{R}$ be a regular time-dependent Lagrangian function. We define the map $Leg : J^1\pi \to T^*E$ by

$$
Leg(j^1_t\phi)(X) = (\Theta_L)_{(j^1_t\phi)}(\tilde{X})
$$

for $j^1_t\phi \in J^1\pi$ and $X \in T_{\phi(t)}E$, where $\tilde{X}$ is a tangent vector at $j^1_t\phi$ such that $(T\pi_{1,0})(\tilde{X}) = X$. In local coordinates we obtain:

$$
Leg(t, q^A, v^A) = (t, q^A, L - v^A\tilde{p}_A, \tilde{p}_A). \tag{21}
$$

Now, if x is a point of E we consider the subspace $(T_v^*E)_x$ of T_x^*E given by

$$(T_v^*E)_x = \{\alpha \in T_x^*E \;/\; i_u\alpha = 0, \forall u \in (V\pi)_x\}.$$

Then, the space $T_v^*E = \bigcup_{x\in E}(T_v^*E)_x$ is a vector subbundle of $\pi_E : T^*E \to E$ of rank 1. We will denote by $J^1\pi^*$ the quotient bundle $J^1\pi^* = T^*E/T_v^*E$. $J^1\pi^*$ is a vector bundle over E of rank n with canonical projection $\pi_{1,0}^* : J^1\pi^* \to E$. $J^1\pi^*$ is also fibred over $\mathbb{R}$ with projection $\pi_1^* = \pi \circ \pi_{1,0}^* : J^1\pi^* \to \mathbb{R}$.

If (t, q^A, p_t, p_A) are local coordinates on T^*E then we have local coordinates (t, q^A, p_t) on T_v^*E and (t, q^A, p_A) on $J^1\pi^*$.

Let $\nu : T^*E \to J^1\pi^*$ be the canonical projection. We denote by $leg : J^1\pi \to J^1\pi^*$ the map $leg = \nu \circ Leg$. Using (21) and the fact that L is regular, we deduce that Leg is an inmersion and that leg is a local diffeomorphism. Assume, for the sake of simplicity, that L is hyper-regular, that is, $leg : J^1\pi \to J^1\pi^*$ is a global diffeomorphism. In such a case, we define a global section $h : J^1\pi^* \to T^*E$ of the projection $\nu : T^*E \to J^1\pi^*$ by $h = Leg \circ leg^{-1}$ (if L is regular we only have local sections of ν). h will be called a Hamiltonian.

If ω_E is the canonical symplectic form on T^*E, we consider on $J^1\pi^*$ the 2-form Ω_h given by $\Omega_h = h^*\omega_E$. A direct computation proves that:

(i) $leg^*\Omega_h = \Omega_L$ and $leg^*\eta_1 = \eta$, where η_1 is the 1-form on $J^1\pi^*$ given by $\eta_1 = (\pi_1^*)^*(\mathrm{d}t)$;

(ii) the pair (Ω_h, η_1) is a cosymplectic structure on $J^1\pi^*$;

(iii) if X_h is the Reeb vector field for (Ω_h, η_1), i.e. $i_{X_h}\Omega_h = 0$, $i_{X_h}\eta_1 = 1$, then ξ_L and X_h are leg-related;

(iv) suppose that in local coordinates

$$h(t, q^A, p_A) = (t, q^A, H(t, q^A, p_A), p_A).$$

Then, the integral curves of X_h satisfy the Hamilton equations

$$\frac{\mathrm{d}q^A}{\mathrm{d}t} = -\frac{\partial H}{\partial p_A} \qquad \frac{\mathrm{d}p_A}{\mathrm{d}t} = \frac{\partial H}{\partial q^A}.$$

Now, suppose that $L : J^1\pi \to \mathbb{R}$ is subjected to the non-holonomic constraints given by the distribution D on E. Since $leg : J^1\pi \to J^1\pi^*$ is a diffeomorphism, we obtain that $\bar{D} = leg(\tilde{D})$ is a submanifold of $J^1\pi^*$ and we can transport the distributions D^{c} and D^{v} from $J^1\pi$ to $J^1\pi^*$. The induced distributions will be denoted by $\bar{D}^{\mathrm{c}}$ and $\bar{D}^{\mathrm{v}}$, respectively. The constrained equations would be

$$i_{\tilde{X}}\Omega_h \in (\bar{D}^{\mathrm{v}})^0 \qquad i_{\tilde{X}}\eta_1 = 1 \qquad \tilde{X} \in \bar{D}^{\mathrm{c}} \tag{22}$$

along $\bar{D}$. We also can transport the distribution S to $J^1\pi^*$ and obtain a distribution $\bar{S}$ on $J^1\pi^*$ along $\bar{D}$. Notice that $\bar{S}$ is locally generated by the vector fields $\bar{Z}_1, \ldots, \bar{Z}_m$, where $\bar{Z}_i$ is the $\pi_{1,0}^*$-vertical vector field on $J^1\pi^*$ defined by

$$i_{\bar{Z}_j}\Omega_h = (leg^{-1})^*(\bar{\mu}_j) \qquad i_{\bar{Z}_j}\eta_1 = 0$$

for all $j \in \{1, \ldots, m\}$. Of course, Z_i and $\bar{Z}_i$ are leg-related.

If the constrained system is regular, $\bar{S}_x \cap T_x\bar{D} = \{0\}$, $\forall x \in \bar{D}$. Proceeding as in the Lagrangian side, we construct an almost product structure $(\bar{\mathcal{P}}, \bar{\mathcal{Q}})$ on $J^1\pi^*$ along $\bar{D}$, which is leg-related with $(\mathcal{P}, \mathcal{Q})$. Then, the vector $\bar{\mathcal{P}}(X_{h/\bar{D}})$ is the unique solution of the constrained Hamilton equations (22). Moreover, since ξ_L and X_h are leg-related we conclude that $\mathcal{P}(\xi_{L/\tilde{D}})$ and $\bar{\mathcal{P}}(X_{h/\bar{D}})$ are leg-related.

If the constrained system is singular, $T_x\tilde{D} \cap S_x \neq 0$ for some point x of $\tilde{D}$. In this case, using the diffeomorphism $leg : J^1\pi \to J^1\pi^*$ we can transport the distribution S_L to $J^1\pi^*$ and obtain a distribution $\bar{S_L}$ on $J^1\pi^*$ along $\bar{D}$. Furthermore, if we apply the algorithm developed in section 9 to equations (22), we obtain a sequence of submanifolds $\bar{D}_i$, where

$$\bar{D}_i = \{x \in \bar{D}_{i-1} / T_x\bar{D}_{i-1} \cap \bar{S}_x \subsetneq T_x\bar{D}_{i-1} \cap (\bar{S_L})_x\} \qquad i > 1$$

and $\bar{D}_1 = \bar{D}$. It is evident that $\bar{D}_i = leg(\tilde{D}_i)$. Thus, both algorithms are related by means of the Legendre transformation $leg : J^1\pi \to J^1\pi^*$, so that if one of them stabilizes at some step k, the other one stabilizes too, and at the same level k.

Again, one can construct an almost product structure along the final constraint submanifold $\bar{D}_f$ such that the projection of the vector field $(X_h)_{/\bar{D}_f}$ gives the dynamics for the constrained system.

11. Constraints defined by connections

Assuming that E is a fibred manifold over a manifold N which also turns out to be a fibred manifold over $\mathbb{R}$, we have the following commutative diagram

$$\begin{tikzcd}
E \arrow[dr, "\rho"] \arrow[dd, "\pi"'] & \\
& N \arrow[dl, "\gamma"] \\
\mathbb{R} &
\end{tikzcd}$$

where π, ρ and γ are surjective submersions such that $\pi = \gamma \circ \rho$. We also assume that a connection Γ on the fibred manifold $\rho : E \longrightarrow N$ is given.

Taking 1-jet prolongations we obtain the following commutative diagram

$$\begin{tikzcd}
J^1\pi \arrow[dr, "j^1\rho"] \arrow[dd, "\pi_{1,0}"'] & \\
& J^1\gamma \arrow[dd, "\gamma_{1,0}"] \\
E \arrow[dr, "\rho"] \arrow[dd, "\pi"'] & \\
& N \arrow[dl, "\gamma"] \\
\mathbb{R} &
\end{tikzcd}$$

where all the arrows again define fibred manifolds. We choose adapted local coordinates (t, q^a, q^i) for the fibred manifold E such that

$$\rho(t, q^a, q^i) = (t, q^a) \qquad \pi(t, q^a, q^i) = t \qquad \gamma(t, q^a) = t.$$

In this section we will consider a Lagrangian function $L : J^1\pi \longrightarrow \mathbb{R}$ subjected to non-holonomic constraints given by the horizontal distribution H of Γ. This means that the only allowable motions are horizontal curves.

We will construct a suitable basis for H. If we denote by X^H the horizontal lift of a vector field X on N to E, we obtain

$$\left(\frac{\partial}{\partial t}\right)^H = \frac{\partial}{\partial t} - \Gamma^i \frac{\partial}{\partial q^i}$$
$$\left(\frac{\partial}{\partial q^a}\right)^H = \frac{\partial}{\partial q^a} - \Gamma^i_a \frac{\partial}{\partial q^i}$$

where $\Gamma^i = \Gamma^i(t, q^b, q^j)$ and $\Gamma^i_a = \Gamma^i_a(t, q^b, q^j)$ are the Christoffel components of the connection Γ.

Thus, we have

$$H = \left\langle \left(\frac{\partial}{\partial t}\right)^H, \left(\frac{\partial}{\partial q^a}\right)^H \right\rangle$$

and

$$\left\{ \left(\frac{\partial}{\partial t}\right)^H, \left(\frac{\partial}{\partial q^a}\right)^H, \frac{\partial}{\partial q^i} \right\}$$

is a local basis of vector fields on $J^1\pi$. A straightforward computation shows that

$$\{\eta = \mathrm{d}t, \eta_a = \mathrm{d}q^a, \eta_i = \mathrm{d}q^i + \Gamma^i_a\, \mathrm{d}q^a + \Gamma^i \mathrm{d}t\}$$

is the local dual basis and, moreover, we have

$$H^0 = \langle \eta_i \rangle.$$

So, the constraint functions have the form

$$v^i + \Gamma^i_a v^a + \Gamma^i = 0.$$

Define the curvature of Γ as the tensor field of type (1,2) on E given by

$$R = \tfrac{1}{2}[\boldsymbol{h}, \boldsymbol{h}]$$

where $\boldsymbol{h}$ is the horizontal projector associated with Γ, and $[\boldsymbol{h}, \boldsymbol{h}]$ is its Nijenhuis tensor (see [20]). Thus,

$$R(\boldsymbol{h}(u_1), \boldsymbol{h}(u_2)) = v([\boldsymbol{h}(u_1), \boldsymbol{h}(u_2)]) \qquad (23)$$
$$R(\boldsymbol{h}(u_1), \boldsymbol{v}(u_2)) = 0 \qquad (24)$$
$$R(\boldsymbol{v}(u_1), \boldsymbol{v}(u_2)) = 0 \qquad (25)$$

for any $u_1, u_2 \in T_x E$, where $\boldsymbol{v} = \mathrm{id} - \boldsymbol{h}$ is the complementary vertical projector. Since

$$\boldsymbol{h}\left(\frac{\partial}{\partial t}\right) = \frac{\partial}{\partial t} - \Gamma^i \frac{\partial}{\partial q^i}$$
$$\boldsymbol{h}\left(\frac{\partial}{\partial q^a}\right) = \frac{\partial}{\partial q^a} - \Gamma^i_a \frac{\partial}{\partial q^i}$$
$$\boldsymbol{h}\left(\frac{\partial}{\partial q^i}\right) = 0$$

we obtain from (23) that

$$R\left(\boldsymbol{h}\left(\frac{\partial}{\partial t}\right), \boldsymbol{h}\left(\frac{\partial}{\partial q^a}\right)\right) = R^i_{0a}\frac{\partial}{\partial q^i} \tag{26}$$

$$R\left(\boldsymbol{h}\left(\frac{\partial}{\partial q^a}\right), \boldsymbol{h}\left(\frac{\partial}{\partial q^b}\right)\right) = R^i_{ab}\frac{\partial}{\partial q^i} \tag{27}$$

where

$$R^i_{0a} = -\frac{\partial \Gamma^i_a}{\partial t} + \frac{\partial \Gamma^i}{\partial q^a} + \Gamma^j\frac{\partial \Gamma^i_a}{\partial q^j} - \Gamma^j_a\frac{\partial \Gamma^i}{\partial q^j}$$

$$R^i_{ab} = -\frac{\partial \Gamma^i_b}{\partial q^a} + \frac{\partial \Gamma^i_a}{\partial q^b} + \Gamma^j_a\frac{\partial \Gamma^i_b}{\partial q^j} - \Gamma^j_b\frac{\partial \Gamma^i_a}{\partial q^j}.$$

According to section 6, the constrained motion equations can be written as follows,

$$i_X\Omega_L \in (H^{\mathrm{v}})^0 \qquad i_X\eta = 1 \qquad X \in H^{\mathrm{c}} \tag{28}$$

along the points of $\tilde{H} = H \cap J^1\pi$. Equations (28) can be equivalently written as

$$i_X\Omega_L = \lambda^i\bar{\eta}_i \qquad i_X\,\mathrm{d}t = 1 \qquad \eta^{\mathrm{c}}_{i\,/J^1\pi}(X) = 0 \qquad \bar{\eta}_i(X) = 0 \tag{29}$$

along the points of $\tilde{H}$.

Now, we will consider a particular kind of constrained system, those called Čaplygin systems.

Definition 11.1. A Čaplygin system is a constrained system given by a regular Lagrangian L on $J^1\pi$ constrained by the horizontal subspaces of a connection Γ in the fibration $\rho : E \longrightarrow N$, such that

$$L((u^H)_{x_1}) = L((u^H)_{x_2}) \tag{30}$$

for any $u \in T_yN$, $y \in N$, $x_1, x_2 \in E$, where $\rho(x_1) = \rho(x_2) = y$, and $\gamma^*(\mathrm{d}t)_y(u) = 1$.

Locally, condition (30) is translated as follows:

$$L(t, q^a, q^i, v^a, -\Gamma^i - v^a\Gamma^i_a) = L(t, q^a, \bar{q}^i, v^a, -\Gamma^i - v^a\Gamma^i_a) \qquad \forall q^i, \bar{q}^i. \tag{31}$$

Remark 11.2. This class of constrained systems were originally considered by Čaplygin [23], and recently studied by Koiller in the autonomous setting [12] (see also [19]). Here, we consider the non-autonomous case.

Condition (30) permits us to define a Lagrangian function L^* on $J^1\gamma$ as follows,

$$L^*(j^1_t\phi) = L((\dot{\phi}(t))^H)$$

for any $j^1_t\phi \in J^1\gamma$. In local coordinates, we deduce from (31) that

$$L^*(t, q^a, v^a) = L(t, q^a, q^i, v^a, -\Gamma^i - v^a\Gamma^i_a)$$

which implies by applying the chain rule that

$$\frac{\partial L}{\partial q^i} - \frac{\partial L}{\partial v^j}\left(\frac{\partial \Gamma^j}{\partial q^i} + v^a\frac{\partial \Gamma^j_a}{\partial q^i}\right) = 0. \tag{32}$$

We write down the constrained motion equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^a}\right) - \frac{\partial L}{\partial q^a} = -\sum_{i=1}^{m} \lambda^i \Gamma_a^i \tag{33}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial q^i} = -\lambda^i \tag{34}$$

where $v^A = \mathrm{d}q^A/\mathrm{d}t$ and λ^i are some Lagrange multipliers to be determined.

From a straightforward but tedious computation, and taking into account (32), (33) and (34), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L^*}{\partial v^a}\right) - \frac{\partial L^*}{\partial q^a} = -\frac{\partial L}{\partial v^i}\left[v^b R_{ab}^i - R_{0a}^i\right]. \tag{35}$$

We can define a 1-form $\alpha_{L,\Gamma}$ along the map $j^1\rho_{/\tilde{H}} : \tilde{H} \to J^1\gamma$ as follows,

$$(\alpha_{L,\Gamma})_{\tilde{x}}(U) = -(\Theta_L)_{\tilde{x}}(\tilde{X}) \tag{36}$$

for $\tilde{x} \in \tilde{H}$ and $U \in T_u(J^1\gamma)$, where $u = j^1\rho(\tilde{x})$ and $\tilde{X} \in T_{\tilde{x}}(J^1\pi)$ is a tangent vector which projects onto the tangent vector $R((u)_x^H, ((T\gamma_{1,0})(U))_x^H)$, with $x = \pi_{1,0}(\tilde{x})$. A direct computation shows that

$$\alpha_{L,\Gamma} = \frac{\partial L}{\partial v^i}\left[v^b R_{ab}^i - R_{0a}^i\right]\theta^a.$$

Next, consider the following equations (along the points of $\tilde{H}$):

$$i_Y \Omega_{L^*} = \alpha_{L,\Gamma} \qquad i_Y \gamma_1^*(\mathrm{d}t) = 1. \tag{37}$$

If L^* is regular, we deduce that there exists a unique vector field ξ^* along the map $j^1\rho_{/\tilde{H}} : \tilde{H} \to J^1\gamma$, that is, $\xi^* : \tilde{H} \to T(J^1\gamma)$ and $\tau_{J^1\gamma} \circ \xi^* = j^1\rho_{/\tilde{H}}$, which verifies (37). Moreover, for each point $\tilde{x} \in \tilde{H}$, we have

$$\tilde{J}(\xi^*(\tilde{x})) = 0$$

where, here, $\tilde{J}$ denotes the vertical endomorphism on $J^1\gamma$. Thus, ξ^* may be viewed as a NSODE along $j^1\rho_{/\tilde{H}}$.

The following theorem relates both mechanical systems.

Theorem 11.3. The constrained Čaplygin system (L, Γ) is regular if and only if L^* is regular. In this case, the solution ξ of the constrained Čaplygin system is related with the solution ξ^* of (37) by projection:

$$T(j^1\rho_{/\tilde{H}})(\xi) = \xi^*$$

that is, the following diagram is commutative:

$$\begin{tikzcd} & T\tilde{H} \arrow[d, "T(j^1\rho_{/\tilde{H}})"] \\ \tilde{H} \arrow[ur, "\xi"] \arrow[r, "\xi^*"'] & T(J^1\gamma) \end{tikzcd}$$

Proof. For a proof of the equivalence between the regularity of the constrained Čaplygin system (L, Γ) and L^* see de León and Martín de Diego [19]. In the quoted paper, the time-independent case was considered, but the proof can be easily adapted for the time-dependent case.

Next, we will prove the second part. The trick of the proof is to define a connection $\bar{\Gamma}$ in the fibration $T(j^1\rho) : J^1\pi \longrightarrow J^1\gamma$ along the submanifold $\tilde{H}$. The horizontal distribution $\bar{H}$ of $\bar{\Gamma}$ is locally spanned by the vector fields

$$\left(\frac{\partial}{\partial t}\right)^{\bar{H}} = \frac{\partial}{\partial t} - \Gamma^i \frac{\partial}{\partial q^i} - \left(\frac{\partial \Gamma^i_a}{\partial t} v^a - \frac{\partial \Gamma^i_a}{\partial q^j}\Gamma^j v^a + \frac{\partial \Gamma^i}{\partial t} - \frac{\partial \Gamma^i}{\partial q^j}\Gamma^j\right)\frac{\partial}{\partial v^i}$$

$$\left(\frac{\partial}{\partial q^a}\right)^{\bar{H}} = \frac{\partial}{\partial q^a} - \Gamma^i_a \frac{\partial}{\partial q^i} - \left[v^b\left(\frac{\partial \Gamma^i_b}{\partial q^a} - \Gamma^j_a \frac{\partial \Gamma^i_b}{\partial q^j}\right) + \left(\frac{\partial \Gamma^i}{\partial q^a} - \frac{\partial \Gamma^i}{\partial q^j}\Gamma^j_a\right)\right]\frac{\partial}{\partial v^i}$$

$$\left(\frac{\partial}{\partial v^a}\right)^{\bar{H}} = \frac{\partial}{\partial v^a} - \Gamma^i_a \frac{\partial}{\partial v^i}.$$

Thus, we obtain a local basis of vector fields on $J^1\pi$ along $\tilde{H}$:

$$\left\{\left(\frac{\partial}{\partial t}\right)^{\bar{H}}, \left(\frac{\partial}{\partial q^a}\right)^{\bar{H}}, \left(\frac{\partial}{\partial v^a}\right)^{\bar{H}}, \frac{\partial}{\partial q^i}, \frac{\partial}{\partial v^i}\right\}.$$

Its dual basis of 1-forms is

$$\{\mathrm{d}t, \mathrm{d}q^a, \mathrm{d}v^a, \bar{\eta}_i, \mathrm{d}((\hat{\eta}_i)_{/J^1\pi})\}.$$

Therefore, the set $\{\bar{\eta}_i, d((\hat{\eta}_i)_{/J^1\pi})\}$ is the annihilator of $\bar{H}$. A simple computation shows that $\bar{H}$ is globally defined along $\tilde{H}$.

If $\bar{\boldsymbol{h}}$ denotes the horizontal projector associated with $\bar{\Gamma}$, we have $\bar{\boldsymbol{h}}^*(\mathrm{d}t) = \mathrm{d}t$, $\bar{\boldsymbol{h}}^*(\mathrm{d}q^a) = \mathrm{d}q^a$, $\bar{\boldsymbol{h}}^*(\mathrm{d}v^a) = \mathrm{d}v^a$, $\bar{\boldsymbol{h}}^*(\bar{\eta}_i) = 0$, and $\bar{\boldsymbol{h}}^*(\mathrm{d}((\hat{\eta}_i)_{/J^1\pi})) = 0$.

Consider the pull-backs of the 1-forms Θ_{L^*} and $\mathrm{d}L^*$ to $J^1\pi$ by means of $T(j^1\rho)$. After a long but straightforward computation we deduce that

$$\bar{\boldsymbol{h}}^*(\Theta_L) = (T(j^1\rho))^*\Theta_{L^*}$$
$$\bar{\boldsymbol{h}}^*(\mathrm{d}L) = (T(j^1\rho))^*\,\mathrm{d}L^*$$

along $\tilde{H}$.

If ξ is the solution of the constrained dynamics on $\tilde{H}$ we have, from lemma 7.8,

$$\mathcal{L}_\xi \Theta_L = \mathrm{d}L - \mathcal{L}_{\mathcal{Q}(\xi_L)}\Theta_L \tag{38}$$

and from lemma IV.4 of [19] and lemma 7.9 we get

$$\begin{aligned}\mathcal{L}_\xi(\bar{\boldsymbol{h}}^*\Theta_L) &= \bar{\boldsymbol{h}}^*(\mathrm{d}L) - \bar{\boldsymbol{h}}^*(\mathcal{L}_{\mathcal{Q}(\xi_L)}\Theta_L) - \bar{\alpha} \\ &= \bar{\boldsymbol{h}}^*(\mathrm{d}L) - \bar{\alpha}\end{aligned}$$

where $\bar{\alpha}$ is the 1-form on $J^1\pi$ along $\tilde{H}$ defined by

$$\bar{\alpha}(Z) = -\Theta_L(\bar{R}(\xi, Z) - \bar{\boldsymbol{h}}([\xi, \bar{\boldsymbol{v}}(Z)]))$$

$\bar{R}$ being the curvature of $\bar{\Gamma}$. Since Θ_L is semibasic and $\bar{\Gamma}$ is a connection in the fibration $T(j^1\rho) : J^1\pi \longrightarrow J^1\gamma$ (along $\tilde{H}$), we deduce that $\Theta_L(\bar{\boldsymbol{h}}([\xi, \bar{\boldsymbol{v}}(Z)])) = 0$, and hence we get

$$\bar{\alpha}(Z) = -\Theta_L(\bar{R}(\xi, Z)).$$

In local coordinates we obtain

$$\bar{\alpha} = \frac{\partial L}{\partial v^i}\left[v^b R^i_{ab} - R^i_{0a}\right]\theta^a.$$

Therefore, we deduce that $\bar{\alpha}$ is precisely the pullback by $j^1\rho_{/\tilde{H}}$ of $\alpha_{L,\Gamma}$.

Thus, we get

$$\mathcal{L}_{\xi}(T(j^1\rho))^*\Theta_{L^*} = (T(j^1\rho))^*(\mathrm{d}L^*) - \bar{\alpha}.$$

Let ξ^* be a vector field along the map $J^1\rho_{/\tilde{H}} : \tilde{H} \longrightarrow J^1\gamma$, and suppose that it is the solution of the equation

$$\mathcal{L}_Y\Theta_{L^*} = \mathrm{d}L^* - \alpha_{L,\Gamma} \qquad i_Y\gamma_1^*(\mathrm{d}t) = 1$$

or, equivalently,

$$i_Y\Omega_{L^*} = \alpha_{L,\Gamma} \qquad i_Y\gamma_1^*(\mathrm{d}t) = 1.$$

Then every vector field $\tilde{Y}$ along $\tilde{H}$ which projects onto ξ^* (that is, $(T(j^1\rho)_{/\tilde{H}})(\tilde{Y}) = \xi^*$) verifies that

$$\mathcal{L}_{\tilde{Y}}(T(j^1\rho))^*\Theta_{L^*} = (T(j^1\rho))^*(\mathrm{d}L^*) - \bar{\alpha} \qquad i_{\tilde{Y}}\eta = 1. \tag{39}$$

Moreover, the horizontal lift of ξ^* with respect to $\bar{\Gamma}$ (i.e., the vector field $\tilde{X}$ such that $\tilde{X}(\tilde{x}) = (\xi^*(\tilde{x})^{\bar{H}})(j^1\rho(\tilde{x}))$, for all $\tilde{x} \in \tilde{H}$) verifies (39). Since ξ also satisfies (39) and $\xi \in \bar{H}$, we deduce that $(\xi^*)^{\bar{H}} = \xi$ and $(T(j^1\rho)_{/\tilde{H}})(\xi) = \xi^*$. □

Remark 11.4. Theorem 11.3 shows that in order to obtain the dynamics of the Čaplygin system (L, Γ) we first reduce the Lagrangian L to a new Lagrangian L^* defined on the reduced phase space. The new system is unconstrained, but subjected to a non-conservative force $\alpha_{L,\Gamma}$. If we solve the dynamics for L^*, we then recover the original dynamics by horizontal lift with respect to the connection $\bar{\Gamma}$. The procedure is close to that known as symplectic reduction procedure.

Corollary 11.5. Let f be a constant of the motion for the non-conservative system $(L^*, \alpha_{L,\Gamma})$, that is, $\xi^*(\tilde{x})(f) = 0$ for all $\tilde{x} \in \tilde{H}$. Then, $(j^1\rho_{/\tilde{H}})^*f$ is a constant of the motion for the Čaplygin system (L, Γ), i.e. $\xi(f) = 0$. Conversely, if g is a projectable function onto $J^1\gamma$ which is a constant of the motion for the Čaplygin system (L, Γ), then its projection is a constant of motion for system $(L^*, \alpha_{L,\Gamma})$.

The last corollary yields a method to obtain constants of the motion for non-holonomic mechanical systems (see also [1, 2]).

Example 11.6. Consider the Lagrangian function L and the distribution D of example 8.5. We have the fibration

$$\begin{array}{rcl} \rho : E = \mathbb{R} \times (\mathbb{R}^2 \times S^1) & \longrightarrow & N = \mathbb{R} \times (\mathbb{R} \times S^1) \\ (t, x, y, \phi) & \longmapsto & (t, x, \phi). \end{array}$$

In the remainder of this example we will restrict ourselves to the open set $\mathbb{R} \times (\mathbb{R}^2 \times U)$, being U the open set of S^1 consisting of the points such that $\cos\phi \neq 0$.

We define a connection Γ in ρ such that the horizontal distribution H is precisely the distribution D. Thus, the distribution H is generated by the vector fields $(\partial/\partial t)$, $(\partial/\partial x) + \tan\phi(\partial/\partial y)$, and $(\partial/\partial\phi)$.

The curvature R of Γ is given by

$$R = -\sec^2\phi\frac{\partial}{\partial y} \otimes (\mathrm{d}x \wedge \mathrm{d}\phi).$$

Since (L, Γ) is a Čaplygin system, we obtain a projected Lagrangian function $L^* : \mathbb{R} \times T(\mathbb{R} \times U) \longrightarrow \mathbb{R}$ given by

$$L^*(t, x, \phi, \dot{x}, \dot{\phi}) = \tfrac{1}{2}(\sec^2\phi)\dot{x}^2 + \tfrac{1}{2}\dot{\phi}^2.$$

Since (L, Γ) is regular (see example 8.5), L^* is regular too, and we have

$$\Theta_{L^*} = \dot{x}\sec^2\phi\,\mathrm{d}x + \dot{\phi}\,\mathrm{d}\phi - \tfrac{1}{2}((\sec^2\phi)\dot{x}^2 + (\dot{\phi})^2)\mathrm{d}t$$

$$\Omega_{L^*} = \mathrm{d}\phi \wedge \mathrm{d}\dot{\phi} + \sec^2\phi\,\mathrm{d}x \wedge \mathrm{d}\dot{x} + 2\sec^2\phi\tan\phi\dot{x}\,\mathrm{d}\dot{x} \wedge \mathrm{d}\phi + (\dot{x})^2\sec^2\phi\tan\phi\,\mathrm{d}\phi \wedge \mathrm{d}t$$
$$+\dot{x}\sec^2\phi\,\mathrm{d}\dot{x} \wedge \mathrm{d}t + \dot{\phi}\,\mathrm{d}\dot{\phi} \wedge \mathrm{d}t.$$

The 1-form $\alpha_{L,\Gamma}$ on $J^1\gamma$ along $\tilde{H}$ is

$$\alpha_{L,\Gamma} = \dot{x}\tan\phi\sec^2\phi(-\dot{\phi}\,\mathrm{d}x + \dot{x}\,\mathrm{d}\phi)$$

which shows that $\alpha_{L,\Gamma}$ is a *bona fide* 1-form on $J^1\gamma$.

The vector field ξ^* which is a solution of the equations

$$i_Y\Omega_{L^*} = \alpha_{L,\Gamma} \qquad i_Y\,\mathrm{d}t = 1$$

is just the projection of ξ onto $J^1\gamma$, namely $T(j^1\rho_{/\tilde{H}})(\xi) = \xi^*$, where ξ is the solution of the constrained system (L, H). Its local expression is

$$\xi^* = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{\phi}\frac{\partial}{\partial \phi} - \dot{x}\dot{\phi}\tan\phi\frac{\partial}{\partial \dot{x}}.$$

Since the function $f = x\dot{\phi} - (\tan\phi)\dot{x}$ is a constant of the motion for ξ^*, from corollary 11.5 we deduce that the pullback $(j^1\rho_{/\tilde{H}})^* f$ is a constant of the motion for the constrained system.

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